# Technical Notes

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## Nonintegrability and Chaos in Unsteady Ideal Fluid Flow

Ambady Suresh*
*Princeton University, Princeton, New Jersey*

### Introduction

THIS Note is concerned with two-dimensional, unsteady, ideal (inviscid, incompressible, irrotational) fluid flow. If the stream function is denoted by $\psi(x,y,t)$, where $(x,y)$ are Cartesian coordinates on the plane, then the fluid trajectories are given by

$$\frac{dx}{dt}=\frac{\partial\psi}{\partial y},\qquad \frac{dy}{dt}=-\frac{\partial\psi}{\partial x} \tag{1}$$

Thus, the flow represents a Hamiltonian system of one degree of freedom with $(x,y)$ as coordinate and momenta, respectively, and $\psi$ the Hamiltonian.

The objective of this Note is to state sufficient conditions on $\psi$ so that solutions to Eqs. (1) display apparently chaotic trajectories. This is inspired by the recent discovery of chaos in few vortex systems by Aref and Novikov, and reviewed recently in Aref.[1] By apparently chaotic, we mean solutions which are fully deterministic, but which have no continuous dependence on initial data. We are able to do this only for a limited class of *almost steady flows*, where the stream function $\psi(x,y,t)$ can be expressed as

$$\psi(x,y,t)=\psi_0(x,y)+\epsilon\psi_1(x,y,t)+\mathcal{O}(\epsilon^2),\quad \epsilon\ll 1$$

$\psi_0(x,y)$ and $\psi_1(x,y,t)$ are required to satisfy the following conditions: 1) $\psi_0(x,y)$ has an interior stagnation point P and a streamline $\Gamma_0$ beginning and ending at P. $\Gamma_0$ is called a homoclinic trajectory. 2) $\psi_1(x,y,t)$ is periodic in $t$.

For this class of flows, conditions under which the small periodic perturbation breaks the integrability of the steady flow by introducing *horseshoes* into the dynamics of the perturbed $(\epsilon\neq 0)$ system will be stated. The motivating example here is the flow due to two-point vortices of unit strength located at $(w(t),0)$ and $(-w(t),0)$, where $w(t)=1+\epsilon\cos\omega t$. The vortices are considered to be "bound vortices" and their oscillatory motion is due to external agencies. In this case,

$$\psi(x,y,t)=(1/2\pi)\ln[\,|z^2-w^2|\,] \tag{2}$$

and $\psi_0$ and $\psi_1$ may be calculated to be

$$\psi_0(x,y)=(1/2\pi)\ln[\,|z^2-1|\,] \tag{3}$$

$$\psi_1(x,y,t)=Re\left[-\frac{1}{2\pi}\frac{2\cos\omega t}{(z^2-1)}\right] \tag{4}$$

where $z=x+iy$. The unperturbed flow, given by $\psi_0(x,y)$, has a stagnation point at the origin, and $\Gamma_0$ the homoclinic streamline is given by $|z^2-1|=1$.

### Conditions for Chaos

For the almost steady flows defined above, Eqs. (1) become identical to the periodically forced nonlinear oscillator, where apparently chaotic solutions have been shown to exist by Melnikov,[2] Arnold,[3] Holmes,[4] and Holmes and Marsden.[5] Due to the formal equivalence of the equations, these results are immediately applicable here.

The method, called the Melnikov function technique, begins by examining the Poincaré map of Eqs. (1), i.e., the map $f:R^2\to R^2$, which takes the point $(x(0),y(0))$ to $(x(T), y(T))$, where $T$ is the period of the Hamiltonian. The origin is a hyperbolic fixed point of this map. Under perturbation, the stable and unstable separatrices through the origin intersect transversally at a point Q, called a homoclinic point. The existence of one such point implies the existence of an infinity of such points. Near each of these points there is an extremely complicated invariant set called a *Smale horseshoe* which possesses orbits of all periods in addition to dense nonperiodic orbits.[6] In addition, the flow cannot admit a real analytic integral in any neighborhood of Q. For details and an excellent account of Melnikov theory, the reader is referred to Ref. 5.

Melnikov[2] derived sufficient conditions for the existence of a homoclinic point Q for the perturbed system, which were later simplified by Arnold.[3] This result is stated in the present context as follows: Let $\psi(x,y,t)$ be an "almost steady flow" satisfying conditions 1 and 2 stated previously. Then the fluid trajectories given by Eqs. (1) are nonintegrable and apparently chaotic if the function

$$M(t_0)=\int_{-\infty}^{\infty}[\psi_0,\psi_1](t-t_0)\,dt \tag{5}$$

has simple zeros, where the Poisson bracket

$$[\psi_0,\psi_1]=\frac{\partial\psi_0}{\partial x}\frac{\partial\psi_1}{\partial y}-\frac{\partial\psi_0}{\partial y}\frac{\partial\psi_1}{\partial x}$$

and the integration is over the unperturbed homoclinic streamline $\Gamma_0$. The function $M(t_0)$ is a measure of the distance between the stable and unstable branches of the separatrix of the perturbed system.

We now apply this condition to the example mentioned earlier. The calculation is easily carried out in complex notation. If we define

$$W_0=\frac{1}{2\pi}\ln(z^2-1),\qquad W_1=-\frac{1}{2\pi}\frac{2\cos\omega t}{(z^2-1)} \tag{6}$$

then the Poisson bracket becomes

$$[\psi_0,\psi_1]=-\mathrm{Im}\frac{dW_0^*}{dz}\frac{dW_1}{dz} \tag{7}$$

where the asterisk stands for complex conjugation.

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*Presently with NL Industries, Houston, Texas.

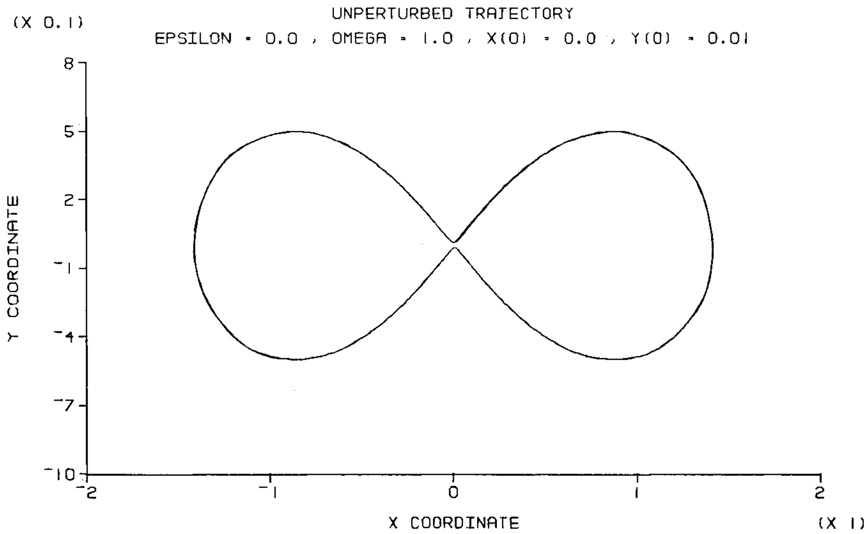


**Fig. 1 Path lines of a fluid particle initially at (0, 0.01) in the presence of unit vortices at (+1, 0) and (−1, 0).**

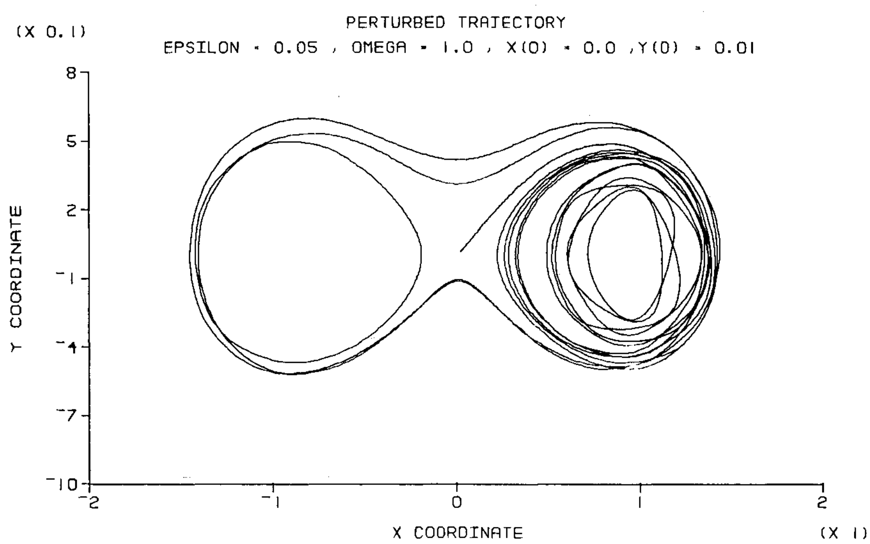


**Fig. 2 Path lines of a fluid particle initially at (0, 0.01) in the presence of unit vortices at ($+w(t)$, 0) and ($-w(t)$, 0) with $w(t)=1+\epsilon\cos\omega t$.**

Using the expressions for $W_0$ and $W_1$, we obtain

$$[\psi_0,\psi_1] = +\frac{4}{\pi^2}\cos\omega t\sin\phi(t)\cos\left(\frac{\phi(t)}{2}\right) \tag{8}$$

on the homoclinic streamline $\Gamma_0$, where $\phi=\arg(z^2-1)$. It now remains to evaluate $\phi(t)$ on $\Gamma_0$. Note that Eqs. (1) with $\epsilon=0$ may be written as

$$\frac{dz^*}{dt}=i\frac{dW_0}{dz}$$

from which we obtain

$$\frac{d\phi}{dt}=-\frac{4}{\pi}\cos\left(\frac{\phi}{2}\right)$$

which has the solution

$$\log\left(\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)\right)=-\frac{2t}{\pi} \tag{9}$$

where we have set $\phi(0)=0$. Equations (8) and (9) are inserted into Eq. (5) and after some simplification, we finally obtain

$$M(t_0)=\frac{16}{\pi^2}\sin(\omega t_0)\int_0^\infty \frac{\sin\omega t\tanh(2t/\pi)}{\cosh^2(2t/\pi)}dt$$

Thus $M(t_0)$ will have simple zeros provided the integral does not vanish. The integral is nonzero for all values of $\omega\neq 0$, $\infty$. Thus the flow is nonintegrable and apparently chaotic for all $\omega\neq 0$, $\infty$.

To confirm these results, Eqs. (1) were numerically integrated with $\psi$ given by Eq. (2). The unperturbed and perturbed trajectories are shown in Figs. 1 and 2, respectively, where the nonintegrability of the latter is apparent. It must be emphasized that the trajectories far from $\Gamma_0$ are relatively unaffected by the perturbation, and the *splatter* is localized around the unperturbed separatrix $\Gamma_0$. The splatter decreases with increasing $\omega$ and this is to be expected, since $M(t_0)\to 0$ as $\omega\to\infty$.

It is perhaps intuitively clear that the far field is relatively unaffected by the periodic motion of the vortices, and one may expect some trajectories in the far field to be closed curves. In fact, a general result called the Kolmogorov-Arnold-Moser (KAM) theorem bears on this question. Briefly, the theorem says that a given orbit is closed if the ratio $\omega_0/\omega$ (where $\omega_0$ is the frequency of the orbit in the unperturbed system and $\omega$ the exciting frequency of the vortices) is "far" from a rational number. This is a vague statement since any number may be approximated by a rational to any degree of accuracy. However, this condition can be precisely stated; the interested reader is referred to Ref. 7 for details. Conversely, it follows that when $\omega_0/\omega$ is a rational number then that orbit is no longer closed under perturbation. This is very reminiscent of linear parametric resonance, where one incurs a loss of stability for certain frequencies of excitation.

## References

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# Effects of Blowing on the Görtler Instability of Boundary Layers


J. M. Floryan*
*The University of Western Ontario, London, Canada*


## Introduction

It is known that Görtler instability may considerably accelerate the laminar-turbulent transition process in boundary layers over concave walls. Therefore, it is of interest to investigate different methods of manipulation of such flows with a view toward eliminating or at least reducing the intensity of this instability. The present analysis deals with the effects of self-similar blowing. It is an extension of an earlier work of Floryan and Saric[1] that dealt only with suction. Self-similar blowing has been considered by Kobayashi[2]; however, the present work is based on a model rationally incorporating the effects of boundary-layer growth[3] and, therefore, the present results represent a considerable improvement over the results of Ref. 2.

## Theory

The linear stability of an incompressible two-dimensional boundary layer is considered. The leading-order approximation for the disturbance equations has the form[3]

$$\beta u + \frac{dv}{d\psi} + \alpha w = 0 \tag{1}$$

$$\beta U u + u\frac{\partial U}{\partial \Phi_l} + \frac{\partial U}{\partial \psi} v + V\frac{du}{d\psi} = \frac{d^2 u}{d\psi^2} - \alpha^2 u \tag{2}$$

$$\beta U v + \frac{\partial V}{\partial \Phi_l} u + \frac{\partial V}{\partial \psi} v + V\frac{dv}{d\psi} + 2G^2 U u$$
$$= -\frac{dp}{d\psi} + \frac{d^2 v}{d\psi^2} - \alpha^2 v \tag{3}$$

$$\beta U w + V\frac{dw}{d\psi} = \alpha p + \frac{d^2 w}{d\psi^2} - \alpha^2 w \tag{4}$$

where $u$, $v$, and $w$ are the disturbance velocity components in the streamwise $\Phi$, normal-to-the-wall $\psi$, and spanwise $z$ directions, respectively; $p$ the pressure disturbance; $U$ and $V$ the streamwise and normal-to-the-wall basic-state velocity components; $\alpha$ the spanwise wavenumber; $\beta$ the spatial growth rate in the streamwise direction; and $\Phi_l = \epsilon\Phi$, where $\epsilon = \nu/U_\infty\delta_r \ll 1$. The boundary-layer thickness $\delta_r$ is defined as $\delta_r = (\nu\tilde{\Phi}/U_\infty)^{1/2}$, where $U_\infty$ is the freestream velocity, $\nu$ the kinematic viscosity, and $\tilde{\Phi}$ the distance from the leading edge.

The Görtler number $G = U_\infty\delta_r/\nu(\delta_r/\mathcal{R})^{1/2}$ is the critical stability parameter. Here $\mathcal{R}$ denotes the radius of curvature of the wall. Equations (1-4), supplemented by the standard no-slip and no-penetration boundary conditions at the wall and conditions describing the decay of disturbances away from the wall,[1] form an eigenvalue problem for $(\alpha,\beta,G)$. For the case with blowing, the disturbances would not necessarily vanish completely at the surface. However, the boundary conditions are approximately valid when the holes or blowing slots are fine enough.[1] The level of self-similar blowing is defined by the blowing parameter $\gamma$, which is defined in Ref. 1.

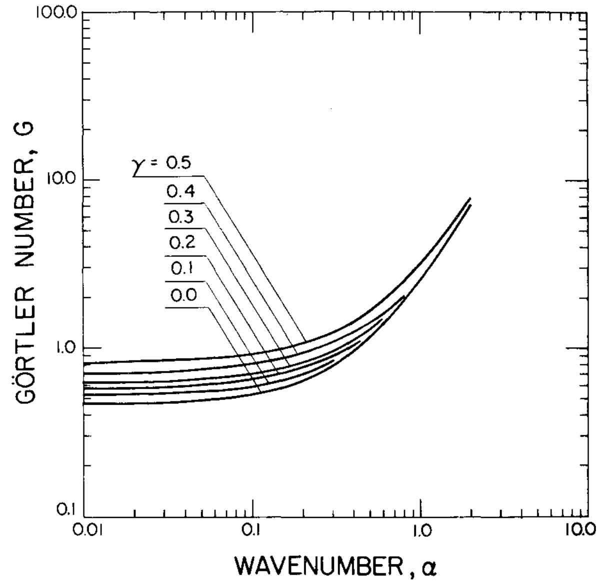


**Fig. 1 Neutral stability curves for the Blasius boundary layer with self-similar blowing.**

## Results

Figure 1 displays neutral stability curves for different levels of self-similar blowing. The results suggest that blowing stabilizes the flow, which is in agreement with Kobayashi.[2] The critical Görtler number varied from $G_{\rm crit} = 0.464$ for the no-blowing case to $G_{\rm crit} = 0.833$ for blowing of $\gamma = 0.5$.

The disturbance growth process[4] is illustrated in Figs. 2-4. Figure 2 shows curves of constant amplification rate for blowing of $\gamma = 0.2$. These curves have an appearance similar to the no-blowing case,[3] except for the small-amplification curves that seem to be more compressed. Figure 3 displays the total amplification of the disturbances that occurred between the neutral curve and a chordwise location corresponding to the Görtler number $G = 20.0$. The total amplification is defined as[1]

$$A = \exp\left[\int_{G_0}^{G} \frac{4}{3}\frac{\beta}{G} dG\right] \tag{5}$$

where $A(G_0) = 1$. Here $A$ denotes the amplitude of the disturbances and subscript $0$ the initial conditions. Each integration begins at the neutral curve and follows the same vortex defined by the constant-dimensional wavelength $\lambda$ downstream. The results, which are presented in terms of the wavelength parameter[1] $\Lambda = U_\infty\lambda/\nu(\lambda/\mathcal{R})^{1/2}$, suggest that,


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*Assistant Professor, Faculty of Engineering Science. Member AIAA.